

# A Cusp in QED at $g = 2$

Johann Rafelski and Lance Labun

Department of Physics, University of Arizona, Tucson, Arizona, 85721 USA

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We explore nonperturbative properties of the dimension-4 QED allowing a gyromagnetic ratio  $g \neq g_D = 2$ . We determine the effective action  $V_{\text{eff}}$  for an arbitrarily strong constant and homogeneous field. Using the external field method, we find a cusp as a function of the gyromagnetic factor  $g$  in a) the QED  $b_0$ -renormalization group coefficient, and in a b) subclass of light-light scattering coefficients obtained in the long wavelength limit expansion. We discuss precision QED results indicating an opportunity for resolution of known theory-experiment disagreements. We show possibility of asymptotic freedom in an Abelian theory for certain domains of  $g$ .

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**Motivation:** No known particle has exactly the Dirac value  $g_D = 2$  of the gyromagnetic ratio  $g$ . Determination of the higher order vacuum fluctuation correction to  $g$  provides the most precise test of Dirac-QED (D-QED), and this test is facing current challenges: 1) There is a three standard deviation discrepancy between theory and experiment involving the directly measured muon gyromagnetic ratio [1, 2], and theoretical result in 8th order  $\alpha^4$  [3]. 2) In a measurement of the muonic hydrogen Lamb shift, a discrepancy between D-QED and experiment is reinterpreted as a revised size of the proton [4, 5], inconsistent by five standard deviations with other available experimental information [6].

In our opinion these recent experimental developments signal need to reexamine the framework of D-QED focusing on the electron-magnetic field interaction. Recall that the D-QED perturbative expansion is for the Dirac value  $g = g_D = 2$ . However, the theory of charged particles interacting with the photon field, which we refer to generically as QED, taken as a stand-alone theory has a point-like electron or muon where in general  $g \neq 2$  due to modifications by other interactions. Perturbative expansion around  $g = g_D$  is not appropriate should this value  $g = g_D$  be a singular point. The aim of present work is to show that there is a singularity at  $g = g_D$ , to study the nature of this singularity, and to lay foundation for a framework allowing exploration of  $|g| > g_D$ .

To achieve these goals we consider the extension to  $g \neq 2$  based on the renormalizable dimension-4 action [7, 8]. We find a singularity at  $g = g_D$  employing the external field method: this means that we study the vacuum properties in presence of external constant and homogeneous electromagnetic fields, integrating out fluctuations of spin-1/2 particles with  $g \neq g_D$ . The resulting effective potential  $V_{\text{eff}}$  is a generalization of the Heisenberg-Euler-Schwinger (HES) effective action [9–13] to arbitrary value of  $g$ . Our result is regular for all  $|g| \leq g_D$  [14].

For  $|g| > g_D$  the HES effective action derived in proper time formulation [14] based on dimension-4 renormalizable QED formulation becomes singular. We propose a natural extension for all values  $|g| > g_D$  which shows that  $g = g_D$  (and other periodic recurrent values) is a cusp point as function of  $g$ . This extension resolves the

known difficulties in the theoretical framework of  $g \neq g_D$  theories [15]. Considering the beta-function and light-light scattering coefficients computed below, we discuss how the results we obtain for  $g \neq g_D$  can be tested by experiment and may impact at the required level the already described two challenges QED faces today.

**Introducing magnetic moment  $|g| \neq g_D$ :** One way to account for  $|g| \neq g_D$  is to complement the Dirac action with an incremental Pauli interaction term  $\delta\mu(\vec{\sigma} \cdot \vec{B} + i\vec{\alpha} \cdot \vec{E}) = \delta\mu\sigma_{\alpha\beta}F^{\alpha\beta}/2$  where  $\vec{E}, \vec{B}$  are the electromagnetic fields,  $F^{\alpha\beta}$  the electromagnetic field strength tensor,  $\sigma_{\alpha\beta} = (i/2)[\gamma_\alpha, \gamma_\beta]$  with  $\gamma_\alpha$  the usual Dirac matrices, and  $\vec{\sigma}$ , and  $\vec{\alpha} = \gamma_5\vec{\sigma}$  are the Pauli-Dirac matrices. This incremental Pauli interaction is a dimension 5 operator,  $[\bar{\psi}\sigma_{\alpha\beta}F^{\alpha\beta}\psi] = L^{-5}$ . The coefficient  $\delta\mu$  consequently has dimension length, which in the case of a composite particle such as the proton is naturally related to the particle size. This Dirac-Pauli (DP) equation has been a popular and effective tool to describe to lowest order the magnetic moment dynamics of a composite particle of finite size, e.g. proton.

A distinct approach to introduce an effective action for  $|g| \neq g_D$  is obtained by adding the full Pauli interaction term to the Klein-Gordon action

$$\mathcal{L} = \bar{\psi} \left[ \Pi^2 - m^2 - \frac{g}{2} \frac{e\sigma_{\alpha\beta}F^{\alpha\beta}}{2} \right] \psi, \quad (1)$$

where  $\Pi_\alpha = i\partial_\alpha + eA_\alpha$ . Note that the dimension of the  $\psi$  field is  $[\psi] = L^{-1}$  and consequently the Pauli interaction is dimension 4. We refer to the study of QED based on Eq. (1) as  $g$ -QED, and the dynamical equation following from Eq. (1) as the Klein-Gordon-Pauli (KGP) equation.  $g$ -QED is the  $s = 1/2$  case in the study of particles of all spins in the Poincaré group framework initiated by Rarita and Schwinger [16]. For recent developments see references in introduction to Ref.[7].

Since there are at least two distinct paths to introduce  $g \neq 2$  corrections into relativistic particle dynamics, the question is in what sense these could be equivalent and if not, which of the two forms is appropriate for study of particle dynamics and/or vacuum structure and under what conditions:

1) The DP approach, involving a dimension-5 operator,

requires new counter terms in each order. This is limiting DP approach to situations in which the physical particle properties are known and vacuum fluctuations need not be considered. Even so, vacuum fluctuations and the effective action  $V_{\text{eff}}$  have been considered in the DP i.e. modified D-QED approach [17–19].

2) In  $g$ -QED the magnetic moment is point-like and consideration of  $g \neq g_D$  does not require a higher dimensioned operator. Therefore the quantum field theory is renormalizable [7, 8], requires a finite number of counter terms, and vacuum fluctuations can be considered in any higher order. It should be remembered that  $g$ -QED constitutes an expansion around  $g = 0$  and not around  $g = \pm 2$  as is the case for DP approach. Properties of the KGP-originating effective action were considered for general spin in Ref. [20], but this work did not recognize the restricted validity domain of the perturbative approach, for spin-1/2  $-2 \leq g \leq 2$ .

3). A recent discussion of quantum field amplitudes with an anomalous moment [21] also arrives at a second-order effective theory, but for a reduced two-component spinor. In view of the derivation and properties of their effective theory, a relation between KGP and DP can at best arise in an infinite order resummation in some specific applications and not in general.

Considering that Eq. (1) is 2nd order in time and has four components, the number of dynamical degrees of freedom present in Eq. (1) is 8. That is, there are twice as many as in usual Dirac theory. For the case  $g = 2$  Eq. (1) can be presented as the square of the operator  $\gamma_5 D$ ,  $D = \gamma_\alpha (i\partial_\alpha + eA^\alpha) - m$  and  $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ ,  $\gamma_5^2 = 1$  is the 5th Dirac matrix. This means that for  $g = 2$  Eq. (1) comprises exact duplication of the Dirac degrees of freedom. For  $g \neq 2$  one must search for a projection restricting the full Hilbert space to the physical states.

Veltman has considered reduction of the number of dynamical components working in a two-component formulation. However, there are unresolved challenges [15] in particular related to self-adjointness of the resulting spectrum and thus conservation of probability in temporal evolution. By individually characterizing states, we will present another resolution of this problem that works in presence of externally applied fields and providing a result within the external field method.

**Eigenvalue-sum periodicity as a function of  $g$ :** Our initial objective is to identify the physics content of the 8 degrees of freedom and separate the Hilbert space into two equal size parts that each individually comprise a complete set of states at a fixed given value of  $g$ . To do so, we consider the Landau-orbit spectrum of the operator in brackets in Eq. (1) in the presence of a constant magnetic field  $\vec{B}$

$$E_n = \pm \sqrt{m^2 + p_z^2 + Q|e\vec{B}|[(2n+1) \mp g/2]}, \quad Q = \pm 1, \quad (2)$$

where  $p_z$  is the one dimensional continuous momentum eigenvalue and  $n$  is the Landau orbit quantum number.

We have made explicit the presence of 8 eigenvalues for each value of  $\vec{B}$ , corresponding to three different possible choices of the signs. There are the usual two roots in Eq. (2), a known feature of relativistic dynamics also seen in the Landau spectrum of the Dirac equation where the negative energy states become positive energy antiparticle ‘hole’ states. The  $\mp$  factor inside the root in Eq. (2) arises from two possible particle spin projections onto magnetic field, corresponding to the spin degeneracy.

Having recalled these usual features, we turn our attention to the new feature and write the eigen energy Eq. (2) in the form

$$K = \frac{E_n^2 - m^2 - p_z^2}{|e\vec{B}|} = Q[(2n+1) \mp g/2], \quad Q = \pm 1. \quad (3)$$

This exhibits a new spectrum duplication related to two possible values of  $Q$ . The quantity  $K$  is shown in the top portion of figure 1 as function of  $g$ . We see that between  $-2 \leq g \leq 2$  there is an exact duplication of the spectrum corresponding to  $Q = 1$  and  $Q = -1$ . The ‘squared’ Dirac operator produces two eigenstate-space copies which can be separated in particular applications. These are two sectors of the Hilbert space with the same physical content, and the  $Q = -1$  eigenvalues can be omitted. Thus for  $-2 \leq g \leq 2$  the effective action is obtained by the usual procedure, and the results have already been presented [14].

For  $|g| > 2$ , new physics content arises for external fields of any strength, including arbitrarily weak. First we note that taking Eq. (2) expression at face-value, naively some eigenstates could have  $E^2 < m^2$ , which implies existence of bound localized states in the presence of a constant magnetic field. Such solutions are not required for completeness and would violate Lorentz symmetry; for these reasons, such states cannot be admitted in the spectrum. This situation differs from the  $m^2 + p_z^2 \rightarrow 0$  limit, in which states having  $K < 0$  signal Nielsen-Olesen instabilities of the conventional vacuum state [22]. In the context of spin-1 charged massless gluons in presence of color magnetic field, this instability has been used to obtain the leading coefficient of QCD renormalization group  $\beta$ -function [23]. In  $g$ -QED for finite mass in the weak field limit, there is no magnetic instability.

To compute the effective action we must define which states contribute to the physical spectral sum. The first step is to accomplish (like for the case  $|g| \leq 2$ ) separation of the Hilbert space into two sectors. We divide the states according to whether  $K \geq 0$  or  $K \leq 0$  and denote the respective sectors  $\mathcal{K}^\pm$ . The limit  $K = 0$  where two states coincide occurs at  $g = 2$  since the KGP operator can be written as exact square of the Dirac operator. This situation recurs with the shift of  $g$  by  $4k, k \in \mathbb{Z}$ . There is no change in the number of states in each of the Hilbert space sectors  $\mathcal{K}^\pm$  as an equal number of single particle states is exchanged between both sectors.

The principle we use to determine which states enter the spectral sum is that there should be no localized

bound states in a constant magnetic field. In the notation just introduced, we require  $K \geq 0$  and the  $\mathcal{K}^+$  sector is chosen as representing the physical spectrum. This is an extension from the regular case  $|g| \leq 2$ , where the usual procedure sums over the  $Q = +1$  states and is equivalent to summing over the  $\mathcal{K}^+$  state space. Seeing as  $K \geq 0$  implies  $E^2 \geq m^2$ , the physics is a continuous extension of the case  $g = 2$ , for which it is proved that  $E^2 \geq m^2$  for arbitrary magnetic fields, i.e. there are no bound states [24].

Looking far outside the principal domain  $-2 \leq g \leq 2$ , we see that relativistic Landau eigenstates cross between  $\mathcal{K}^\pm$  at each  $g_k = 2 + 4k, k \in \mathbb{Z}$ . As the graphic representation top frame of Fig. 1 shows, for each of the Hilbert space sectors  $\mathcal{K}^\pm$  we have periodicity of the Landau levels a function of  $g$ . Therefore, the sum  $\sum_n E_n$  over  $\mathcal{K}^+$  leading to the real part of  $V_{\text{eff}}(\vec{B}^2)$ , is a periodic function of  $g$ , a result we will find explicitly. This periodicity does not apply to individual Landau eigenvalues as is seen in Eq. (2). In computation of vacuum fluctuations the truncation of the Landau eigenstate  $n$ -sum to any finite value breaks the periodicity as well.

The choice of  $\mathcal{K}^+$  as the physical state space has clear advantages and resolves the challenges encountered by Veltman [15]: In addition to maintaining self-adjointness of the KGP system, it makes the quantum field theories based on semi-spaces  $\mathcal{K}^\pm$  each individually unitary, because the number of states is conserved in transiting through the singular points e.g at  $|g| = 2$ , and for  $|g| > 2$  we omit the localized solutions. Moreover, our proposal makes the spectrum and by extension the quantum theory a continuous and analytic extension from the domain  $|g| \leq 2$ . Our approach preserves translation invariance of the vacuum, which would be broken by any localized bound states in the constant-field-filled vacuum. It is critical to note that had we separated the sectors along the sign of  $Q$ , the contents of the theory would be different for  $|g| > 2$  and unitarity would be violated since the ‘wrong’ levels would be included in the physical half-space.

**Effective action for  $|g| \leq 2$ :** We briefly summarize results for  $|g| \leq 2$  [14], as these are needed to understand the present case of  $|g| > 2$ . For constant fields the effective action is manifestly covariant, and can be written as a function of the Lorentz-invariant field-like quantities  $a, b$

$$b^2 - a^2 = \vec{B}^2 - \vec{E}^2 = \frac{1}{2} F_{\alpha\beta} F^{\alpha\beta} \equiv 2S, \quad (4)$$

$$(ab)^2 = (\vec{E} \cdot \vec{B})^2 = \left( \frac{1}{8} F^{\alpha\beta} \epsilon_{\alpha\beta\kappa\lambda} F^{\kappa\lambda} \right)^2 \equiv \mathcal{P}^2, \quad (5)$$

where  $\pm a$  are electric-field-like and  $\pm ib$  are the magnetic-field-like eigenvalues of  $F^{\alpha\beta}$ .  $a$  is considered electric-like because  $a \rightarrow |\vec{E}|$  on taking the limit  $b \rightarrow 0$ , and similarly  $b \rightarrow |\vec{B}|$  in the limit  $a \rightarrow 0$ .

The Schwinger-Fock proper time method [11] to evaluate the effective action exploits properties of the ‘squared’

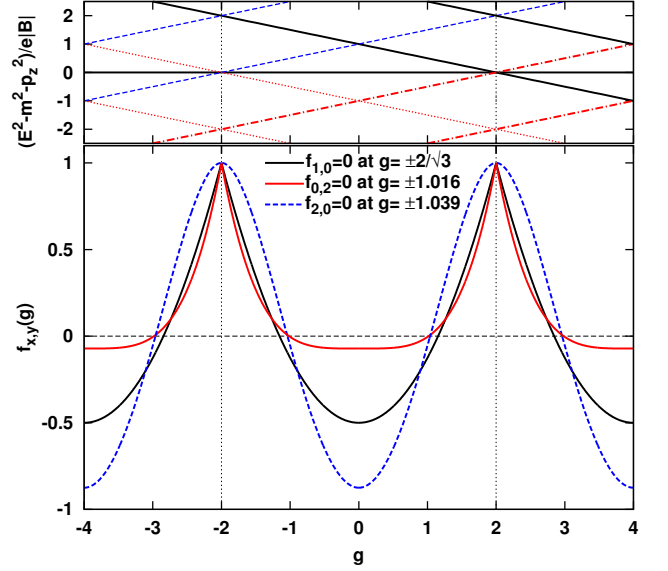


FIG. 1: Top: Squared eigenvalues Eq. (2) of KGP in magnetic field; the solid and (blue) dashed lines are for  $Q = +1$ , and respectively  $(-)$  and  $(+)$  spin eigenvalue; the dotted and dash-dotted (red) lines are for  $Q = -1$ , and respectively  $(+)$  and  $(-)$  spin eigenvalue. Bottom: coefficient functions:  $f_{1,0}(g)$  as defined in Eq. (12) and  $f_{0,2}$  and  $f_{2,0}$  as defined in Eq. (13). Two full periods are shown. The values of  $g$  where the sign of the functions  $f_{i,j}$  changes is indicated.

Dirac equation and thus it can be used to study arbitrary value of  $g$ . The effective action can be written in form

$$V_{\text{eff}} = \frac{1}{8\pi^2} \int_0^\infty \frac{du}{u^3} e^{-i(m^2 - i\epsilon)u} F(eau, ebu, \frac{g}{2}). \quad (6)$$

For  $g = 0, 2$  the proper time integrand  $F(eau, ebu, g)$  was reviewed in Ref. [13]. The generalization throughout the interval  $|g| \leq 2$  is accomplished by inserting into Schwinger's Eq. (2.33) in the last term a co-factor  $g/2$  leading to [14].

$$F(x, y, \frac{g}{2}) = \frac{x \cosh(\frac{g}{2}x)}{\sinh x} \frac{y \cos(\frac{g}{2}y)}{\sin y} - 1, \quad \left| \frac{g}{2} \right| \leq 1. \quad (7)$$

The subtraction  $-1$  in Eq.(7) removes the field-independent constant. The logarithmically divergent charge renormalization term is isolated and discussed below. Note that Eq. (6) would be divergent for  $|g| > 2$  if Eq. (7) were to be used in this domain.

**Effective action for  $|g| > 2$ :** To extend Eq. (7) to  $|g| > 2$ , we consider in more detail the eigenvalue summation method we introduced above, following the work of Heisenberg and Euler [9] and Weisskopf [10]. The mathematical tool used was the L. Euler summation formula, leading to the Bernoulli functions  $B_{2k}(x)$  and Bernoulli numbers  $\mathcal{B}_{2k} \equiv B_{2k}(0)$ . The sum of the Landau energies Eq. (2) involves the form  $\sum_n f(x+n)$ . L. Euler developed the technique for such sums, which manifest an integer

shift symmetry in the variable  $x \rightarrow x + n'$  [25, 26]. Due to this shift symmetry, the Bernoulli functions  $B_{2k}(x)$  that arise in the context of L. Euler summation of Landau energies  $E_n$ , Eq. (2) are the *periodic* Bernoulli functions, given by the Fourier series [27]

$$\tilde{B}_{2k}(t) = (-1)^{k-1} \frac{(2k)!}{2^{2k-1}} \sum_{n=1}^{\infty} \frac{\cos(2\pi nt)}{(n\pi)^{2k}}, \quad (8)$$

(here only needed for an even value of index,  $2k$ ). In the unit interval,  $0 \leq t \leq 1$ , the periodic Bernoulli functions are equal to the Bernoulli polynomials, e.g.  $\tilde{B}_2(t) = B_2(t) = t^2 - t + 1/6$ ,  $0 \leq t \leq 1$ . Outside

the unit interval, the periodic Bernoulli functions Eq. (8)  $\tilde{B}_{2k}(t)$  repeat the polynomials' behavior on  $0 \leq t \leq 1$  in each subsequent period.

Dividing the Landau energies by  $2|e\vec{B}|$  to make the coefficient of  $n$  unity, we see that  $t \rightarrow g/4 + 1/2$  and hence we recognize that the periodic Bernoulli functions with argument  $t = g/4 + 1/2$  appears in the effective action, arising from the summation of eigenvalues. The explicit representation of the argument of Eq. (6) in terms of Bernoulli functions is arrived at employing the analytic transformation of the integrand of Eq. (6) [28, 29].

$$\begin{aligned} F(x, y, \frac{g}{2}) &= (x^2 - y^2) 2 \sum_{n=1}^{\infty} \frac{\cos n\pi(\frac{g}{2} + 1)}{(n\pi)^2} + (x^2 - y^2)^2 2 \sum_{n=1}^{\infty} \frac{\cos n\pi(\frac{g}{2} + 1)}{(n\pi)^4} \\ &+ (xy)^2 4 \left( \sum_{n=1}^{\infty} \frac{\cos n\pi(\frac{g}{2} + 1)}{(n\pi)^4} - 3 \left( \sum_{n=1}^{\infty} \frac{\cos n\pi(\frac{g}{2} + 1)}{(n\pi)^2} \right)^2 \right) + F_6(x, y, \frac{g}{2}), \end{aligned} \quad (9)$$

where we separated the lowest powers of fields from an exact remainder function

$$F_6(x, y, \frac{g}{2}) = y^2 f_6(y^2, \frac{g}{2}) - x^2 f_6(-x^2, \frac{g}{2}) - f_6(-x^2, \frac{g}{2}) f_6(y^2, \frac{g}{2}), \quad f_6(y^2, \frac{g}{2}) = 2y^4 \sum_{n=1}^{\infty} \frac{\cos n\pi(\frac{g}{2} + 1)}{(n\pi)^4 (y^2 + (n\pi)^2)}. \quad (10)$$

The new mathematical element on the RHS in Eq. (9) is that to assure the necessary periodicity we introduced in accordance with Eq. (8) a series of Bernoulli functions with  $t = g/4 + 1/2$ . We note that the right hand side of Eq. (9) agrees exactly with the known expansion [29] in the domain of Eq. (7)  $|g| \leq 2$ . This expression provides an analytical continuation into the domain  $|g| > 2$  having the periodicity property of the effective action identified in study of the full set of eigenvalues.

Each term in Eq. (9) produces a well-defined result for all  $g$  upon performing the proper time integral Eq. (6). The form Eq. (9) is thus a unique and convergent extension to  $|g| > 2$  determined by the Euler summation of the eigenvalues Eq. (2). Even after the removal of the charge renormalization subtraction term (first term on RHS of Eq. (9)) the remainder of the effective action is manifestly periodic in  $g$  but not in  $e$ .

**Nonperturbative in  $g$  renormalization group  $\beta$  function:** The first non constant term on the right hand side of Eq. (9) proportional to  $a^2 - b^2$  isolates the logarithmically divergent one-loop  $\mathcal{O}(\alpha)$   $V_{\text{eff}}$  subtraction required for charge renormalization. The coefficient of this term is related to the  $\beta$ -function coefficient  $b_0$  as is discussed e.g. in section 5.1 in Ref. [13]. The two next terms  $(a^2 - b^2)^2$  and  $(ab)^2$  correspond to lowest order effective field-field interaction potentials describing light-light scattering. Setting in the remainder denominator on RHS of Eq. (10)  $y = 0$  produces next term in the

expansion, etc.

We now consider explicitly the running of the coupling constant  $\alpha$  within the  $g$ -QED loop expansion of the  $\beta$ -renormalization function

$$\beta \equiv \mu \frac{\partial \alpha}{\partial \mu}, \quad \beta(\alpha) = -\frac{b_0}{2\pi} \alpha^2 + \frac{b_1}{8\pi^2} \alpha^3 + \dots \quad (11)$$

The first sum in Eq. (9), for  $g = 2$ ,  $\sum_{n=1}^{\infty} 1/(\pi n)^2 = 1/6$  and implies the value of  $b_0 = -4/3$ , where factor 4 indicates the 4 components of spin-1/2 particle. For arbitrary  $g$ ,  $b_0(g)$  is obtained using Eq. (8) to identify this sum as  $\tilde{B}_2(g/4 + 1/2)$ . The character of this function is manifest by reconnecting periodic domains of the familiar Bernoulli polynomial  $B_2(t) = t^2 - t + 1/6$  and the resulting  $b_0(g)$  coefficient is given in each domain  $g \in [g_{k-1}, g_k]$

$$b_0 = -\frac{4}{3} f_{1,0}(g) = -\frac{4}{3} \left( \frac{3}{8} (g - 4k)^2 - \frac{1}{2} \right), \quad (12)$$

where  $f_{1,0}(g)$  is shown in bottom frame of Fig. 1. The subscripts of  $f_{i,j}$  indicate the powers of the Lorentz invariants in polynomial expansion  $f_{i,j} \mathcal{S}^i \mathcal{P}^j$  in Eq. (9). We see in Fig. 1 that as a function of  $g$ , the Dirac value  $g_D = \pm 2$  is an upper cusp point with  $f_{1,0}(g) \leq f_{1,0}(2) = 1$ . For clarity, two periods are shown in Fig. 1.

Note that our result arises from the KGP equation applying a nonperturbative method in  $g$  to one loop expansion. This approach is necessary in order to obtain the behavior of the  $\beta$ -function for  $|g| > 2$ . At  $g = \pm 2$  we find the unexpected cusp. This feature is missing in perturbative consideration of  $\beta(g)$  at one loop level which



produces the same functional dependence on  $g$  as seen in Eq. (12) setting  $k = 0$ . As our study shows, a perturbative expansion around  $g = 0$  has a finite convergence interval  $|g| \leq 2$ .

The following implications for  $g$ -QED of the properties of the renormalization group coefficient  $b_0(g)$  are noteworthy – we address in the following discussion the range of values of  $g$  shown in Fig. 1:

- 1.) We recognize  $g$  is an independent ‘large’ coupling constant. The first-order D-QED expands around  $g = \pm 2$ , which points are identified as being non-analytic in the  $g$ -QED framework.
- 2.) For any value of  $g$  not at the cusp the magnitude  $|b_0|$  decreases (and thus the speed of ‘running’ decreases) compared to its value at  $g = 2$ . Considering that the coefficient of the magnetic spin term in Eq. (1) is dimensionless there is no new scale appearing in association with  $g$ .
- 3.) The presence of the cusp in  $b_0$  implies that the running coupling of  $g$ -QED,  $\alpha(g)$  comprises the cusp as well.
- 4.) A cross check and confirmation of our result for  $b_0(g)$  is obtained in perturbative domain considering the limit  $g \rightarrow 0$  where  $b_0(g \rightarrow 0)$  differs only by a minus sign and the number of degrees of freedom from the known behavior of scalar particle ‘QED’. The minus sign is due to the commutation relation needed in closing the fermion trace in loops, whereas it is absent in scalar boson loops.
- 5.) In the principal domain  $|g| \leq 2$  the functional dependence on  $g$  we find agrees with the result Eqs. (53–57) seen in Ref. [8]. Specifically, the leading term for large  $q^2$  of the vacuum polarization function, evaluated within the framework of  $g$ -QED is  $-\alpha b_0(g)/(2\pi) \ln(-q^2/m^2)$ , seen explicitly in Eq. (55) of Ref. [8].
- 6.) As the above limit shows, for a range of appropriate gyromagnetic moment values  $g$  (including  $g = 0$ )  $b_0(g) > 0$  is *possible*. This produces asymptotic freedom behavior for fermions interacting alone with an Abelian charge. The switch between the infrared stable and the asymptotically free behavior occurs in the principal  $g$ -domain twice, at  $g = \pm 2/\sqrt{3} = \pm 1.155$  and continues periodically e.g. for  $g = 4 - 2/\sqrt{3} = 2.845$ . This mechanism of asymptotic freedom generation by  $g$ -driven sign reversal is implicit in Eq. (56) of Ref. [8] (valid in principal domain  $|g| \leq 2$ ), but the new mechanism allowing Abelian confinement has not been recognized there. The values of  $g$  where the sign of the functions  $f_{i,j}$  changes is indicated in Fig. 1, up to periodic recurrence.

**Light-light scattering as function of  $g$ :** We find that the cusp at  $|g| = 2$  reappears in a directly observable phenomenon inherent in the Heisenberg-Euler action, the light by light scattering. For the general case of both electric and magnetic fields present, using Eq. (9) we find up to fourth order in the fields

$$V_{\text{eff}} \simeq \frac{\alpha}{2\pi} \frac{e^2}{m^4} \left( \frac{f_{2,0}}{45} \mathcal{S}^2 + \frac{7f_{0,2}}{45} \mathcal{P}^2 \right) \quad (13)$$

$$f_{2,0}(g) = -120\tilde{B}_4(g/4 + 1/2) \quad (13a)$$

$$= -\frac{15(g-4k)^4}{32} + \frac{15(g-4k)^2}{4} - \frac{7}{2} \quad (13b)$$

$$f_{0,2}(g) = -\frac{60}{7} \left[ \tilde{B}_4 \left( \frac{g}{4} + \frac{1}{2} \right) - 3\tilde{B}_2^2 \left( \frac{g}{4} + \frac{1}{2} \right) \right] \quad (13c)$$

$$= \frac{15(g-4k)^4}{224} - \frac{1}{14} \quad (13d)$$

where both  $f_{2,0}$  and  $f_{0,2}$  are normalized to  $g = 2$  values and presented in Fig. 1.  $f_{0,2}$  includes a product of two Bernoulli functions with cusp and so has a steeper cusp. Importantly  $f_{0,2}$  enters the  $\mathcal{P}^2$  term which one actually measures in laser light scattering off a magnetic field [30, 31]. In general, our finding is that all  $f_{i,j}(g)$  for  $j > 0$  have cusps at  $g = 2$  whereas all  $f_{i,0}(g)$ ,  $i > 1$  are continuous and differentiable at  $g = 2$ , being proportional to higher order  $> 2$  Bernoulli functions that have vanishing derivatives at  $g = 2$ . Thus only coefficients of terms involving powers of the pseudo scalar field invariant  $\mathcal{P}^2 = (\vec{E} \cdot \vec{B})^2$  display cusps at  $g = 2$ .

**Discussion:** We found new physics arising for  $|g| > 2$  for arbitrarily weak fields in  $g$ -QED. We proposed a new eigenstate sorting based on sign of  $K$ , Eq. (3), leading to a self-adjoint theory that retains Poincaré symmetry and contains a complete set of particle-antiparticle states, and thus preserves probability in time evolution and analyticity as function of  $g$ , up to a countable set of singular points.

While Eq. (7) is an analytic function of  $g$ , the integral of Eq. (7) with the proper time weight Eq. (6) does not exist for  $|g| > 2$ . Thus a naive extension of HES effective action to  $|g| > 2$  is not possible. This parallels the observation that the Klein-Gordon-Pauli operator Eq. (1) is not self-adjoint for  $|g| > 2$ . We have presented a careful study of how the eigenstate level crossing can be recognized and states assigned to half-spaces of the full Hilbert space, leading to a natural self-adjoint extension and a valid theoretical  $g$ -QED framework for  $|g| > 2$ . The cusp and related nonperturbative in  $g$  effects arise from implementation of the self-adjoint extension described. The origin of the cusp is in the periodic crossing of eigenenergies in the spectrum of Landau eigenstates seen in upper section of Fig. 1 showing the quantity  $K$ , Eq. (3).

The top frame of Fig. 1 illustrates the case of a (weak) magnetic field only, a similar result is obtained for the case of an electric field only, leading to the unique form Eq. (9). Within this expression we have shown cusps at  $g = g_D$  for two physical quantities computed for arbitrary  $g$ :

- The renormalization group coefficient  $b_0$  proportional to function  $f_{1,0}$ , see Fig. 1;
- The light-by-light scattering in the long-wavelength limit comprising a smooth function  $f_{2,0}$ , and for the term  $(\vec{E} \cdot \vec{B})^2$  the cusp function  $f_{0,2}$ , see Fig. 1.

These nonperturbative in  $g$  one-loop results require infinitely many contributions from eigenvalues Eq. (2), and for this reason the effects predicted here are only visible in phenomena arising from vacuum fluctuations, such as the  $\beta$ -function and effective light-light interactions. We have checked that these results can be arrived at directly by the method of  $\zeta$ -function regularization following Weisskopf [10].

Our results agree in the fundamental domain  $-2 \leq g \leq 2$  with earlier perturbative work: the functional dependence on  $g$  is explicit and the same for the vacuum polarization as had been obtained in Ref. [8] in Eq. (56). We have shown by explicit computation that an expansion around  $g = 0$  is valid for  $|g| \leq 2$  only.

**Conclusions and outlook:** Difficulties of D-QED as a stand-alone theory have been known for some time, beginning with the work of G. Källén [32], and perturbative D-QED is believed by many to be semi-convergent only. Exploration of  $g \neq 2$  in a renormalizable theory requires the dimension-4  $g$ -QED based on KGP equation. However,  $g$ -QED has to begin with 8 degrees of freedom and appropriate division into two half-Hilbert spaces is required. Restriction to the usual Dirac-like 4 degrees of freedom is difficult, as a theory with  $g \neq 2$  is in general not unitary [15]. We resolved this problem, and from the solution we discovered that the Dirac value  $g = g_D = 2$  is a cusp point of the effective action  $V_{\text{eff}}$ , Eq. (6) evaluated in renormalizable  $g$ -QED approach.

This finding implies that the D-QED expansion around  $g = g_D$  could be incomplete at sufficiently high order. To see the problem, imagine that we partially resum  $g - 2$  diagrams with Dyson-Schwinger method finding an effective electron with  $g > 2$ . In the next step we want to compute the vacuum polarization inserts in other  $g - 2$  diagrams. Attempts in D-QED framework will encounter new divergences as the  $g - 2$  correction is dimension-5 op-

erator. On the other hand, we can accomplish this task in  $g$ -QED: we use the non-perturbative in  $g$  renormalization group coefficient  $b_0$  to characterize the vacuum polarization loop insert and there are no new divergences. However, the result contains the cusp, and thus is different from the finite order perturbative expansion of D-QED.

We believe that the higher order vacuum polarization modification we described would be most visible for the Lamb shift of muonic hydrogen which is dominated by the vacuum polarization [33–35]. Similarly, we expect that our light-light scattering cusp modifies the corresponding contribution to the muon  $g - 2$ . We discussed in separate work how the top-quark loop modifies the two photon [36] and two gluon [37] decay of the Higgs. Attempts to evaluate these results using D-QED methods, that is a  $g - 2$  non-renormalizable extension, would fail in evaluation of the loop integrals which in general are divergent in this case.

Our study shows how a complete theory of a point-like fermion with  $|g| > 2$  can be constructed within  $g$ -QED in order to allow dynamical description of real world spin-1/2 particles. We have obtained the HES effective potential for an elementary particle with gyromagnetic ratio  $g \neq 2$  nonperturbatively in  $g$ , see Eq. (6) and Eq. (9). We demonstrated a cusp as function of  $g$  at the Dirac value  $g = g_D = 2$ . We have shown how this cusp enters the  $\beta$ -function and  $(\vec{E} \cdot \vec{B})^{2n}$  terms of light-light scattering. An interesting theoretical consequence is the possibility of asymptotic freedom in an Abelian theory with anomalous magnetic moment originating in the reversal in sign of the renormalization group coefficient  $b_0$  for  $g$  in specific domains much different from  $g = 2$ .

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